Available online at www.sciencedirect.com

SCIENCE DIRECT®

International Journal of Heat and Mass Transfer 47 (2004) 1555–1559

International Journal of HEAT and MASS TRANSFER

www.elsevier.com/locate/ijhmt

Technical Note

On transient heat conduction in a one-dimensional composite slab

Yuzhi Sun, Indrek S. Wichman *

Department of Mechanical Engineering, Michigan State University, 2555 Engineering building, East Lansing, MI 48824-1226, USA

Received 11 June 2003; received in revised form 6 September 2003

Abstract

A theoretical solution is presented of a problem of transient heat conduction in a one-dimensional three-layer composite slab. A full series solution for impulsive heating is found by employing a 'natural' orthogonal relationship between the eigenfunctions. The eigenfunction expansion solution is compared with a finite difference numerical solution. Based on a previous analysis of the two-layer problem, and the present three-layer problem, a conjectured partial solution for an n-layer composite slab is given.

Keywords: Composite materials; Conduction; Transient; One-dimensional

1. Introduction

The problem of multi-layer heat conduction has been extensively studied [1–3]. The book by Özisik [1] contains a detailed review (in Chapter 8) of one-dimensional composite media, focusing on orthogonal expansions, Green's functions and Laplace transform techniques. Recently, de Monte [4] has developed a solution for the two-layer problem using a 'natural' eigenfunction expansion method. He also provides a detailed and well-written introduction in which various solution methods are described and compared. The global (or 'natural') eigenfunction expansion method has the advantage of making "the solution consistent with the physical reality of the problem" because the "transient response of (the) solid to changes in the outer boundary conditions is strictly linked to the thermal diffusivity" [4]. The method is both "efficient" [5] and "simple" [5] in the sense that the problem formulation is intuitive and its solution by eigenfunction expansions is straightforward.

As part of a funded NASA microgravity combustion research project dealing with flamelet spread over thin samples of solid fuels (in the presence of a nearby cold "backing" or "substrate"), we generalized this solution method to the three-layer problem. Our three-layer model describes the simplified case in which all heat transfer occurs only by conduction (no convection, no radiation, no heat generation [6], no combustion [7]). Based on the symmetries observed in the two [4] and three-layer problems, we formulate a conjectured n-layer solution for the one-dimension multi-layer slab. Comparisons between the straightforward numerical solution and the eigenfunction expansion solution are made.

Although the straightforward finite-difference numerical solution is easily implemented there may be cases, particularly in problems involving contact resistance and other limiting behaviors (for example, one slab layer is extremely thin or consists of a material very different from the others), where it is advantageous to have an analytical solution.

2. The dimensionless model problem

Consider a composite slab consisting of three parallel layers as shown in Fig. 1. Let k_1, k_2, and k_3 be thermal

* Corresponding author. Tel.: +1-517-353-9180; fax: +1-517-353-1750.

E-mail addresses: sunyuzhi@egr.msu.edu (Y. Sun), wichman@egr.msu.edu (I.S. Wichman).

0017-9310/$ - see front matter © 2003 Elsevier Ltd. All rights reserved.
doi:10.1016/j.ijheatmasstransfer.2003.09.011

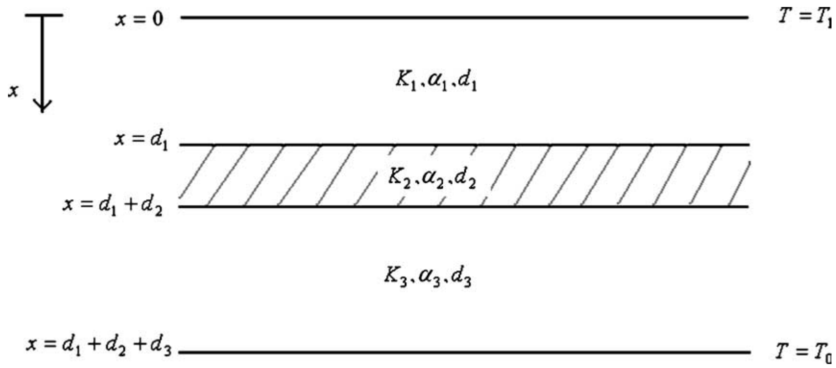

Fig. 1. The three-layer composite slab showing the imposed temperatures on the two sides at the orientation of the coordinate system. The contact between the interfaces is assumed to be thermally perfect, meaning continuity of T and $\partial T/\partial x$ at $x = d_1$ and $x = d_1 + d_2$.

conductivities, α_1, α_2, and α_3 be the thermal diffusivities and d_1, d_2, and d_3 be the thickness of the first, second and third layers, respectively. Initially ($t = 0$) the three-region plate has a specific, uniform temperature T_0. At time $t = 0$ the composite slab is suddenly heated such that the temperatures after $t = 0$ are fixed as T_0 and T_1 at the bottom and top surfaces, respectively. Here T_1 represents the "flame" temperature while T_0 represents the cold "substrate" temperature. The middle region of thickness d_2 is the heated sample. At the two interfaces ($x = d_1, x = d_1 + d_2$), we assume that perfect thermal contact conditions are satisfied. We assume also that the thermal conductivity and the thermal diffusivity are temperature independent and uniform within each layer. In a rectangular coordinate system the mathematical model in dimensionless form is written as follows:

$$\frac{\partial\theta}{\partial\tau} = \delta_i \frac{\partial^2\theta}{\partial\xi_i^2} \text{ (in the } i\text{th region)} \tag{1a}$$

$$\theta|_{\xi_1=0} = 1, \quad \theta|_{\xi_3=1} = 0 \text{ (boundary conditions)} \tag{1b}$$

$$\theta|_{\xi_1\to1^-} = \theta|_{\xi_2\to0^+} \text{ (continuity of temperature)} \tag{1c}$$

$$-\kappa_1 \frac{\partial\theta}{\partial\xi_1}\bigg|_{\xi_1\to1^-} = -\kappa_2 \frac{\partial\theta}{\partial\xi_2}\bigg|_{\xi_2\to0^+} \text{ (continuity of flux)} \tag{1d}$$

$$\theta|_{\xi_2\to1^-} = \theta|_{\xi_3\to0^+} \text{ (continuity of temperature)} \tag{1e}$$

$$-\kappa_2 \frac{\partial\theta}{\partial\xi_2}\bigg|_{\xi_2\to1^-} = -\kappa_3 \frac{\partial\theta}{\partial\xi_3}\bigg|_{\xi_3\to0^+} \text{ (continuity of flux)} \tag{1f}$$

$$\theta|_{\tau=0} = 0 \text{ (initial condition).} \tag{1g}$$

Here, $\theta = (T - T_0)/(T_1 - T_0)$, $\xi_1 = x/d_1$, $\xi_2 = (x - d_1)/d_2$, $\xi_3 = (x - (d_1 + d_2))/d_3$, $\tau = t/t_0$, $\delta_i = \alpha_i t_0/(d_i^2)$, $\kappa_i = k_i/d_i$ for $i = 1, 2, 3$. Here t_0 is a reference time. We also note that κ_1, κ_2, κ_3 are not dimensionless but their ratios (κ_1/κ_2, etc.) are dimensionless. As will be seen later (Section 5), writing the equations in this form makes certain mathematical symmetries immediately obvious.

3. Series solution

We decompose the dimensionless temperature θ into two parts as

$$\theta(\xi_i, \tau) = \psi(\xi_i) - \phi(\xi_i, \tau) \tag{2}$$

Here, $\psi(\xi_i)$ is the steady solution [8], given by

$$\left.\begin{array}{ll} \psi = 1 - (\Delta\theta)_1\xi_1 & \text{in } \xi_1 \in (0,1) \\ \psi = 1 - (\Delta\theta)_1 - (\Delta\theta)_2\xi_2 & \text{in } \xi_2 \in (0,1) \\ \psi = 1 - (\Delta\theta)_1 - (\Delta\theta)_2 - (\Delta\theta)_3\xi_3 & \text{in } \xi_3 \in (0,1) \end{array}\right\} \tag{3}$$

where, $(\Delta\theta)_i = (1/\kappa_i)/(1/\kappa_1 + 1/k_2 + 1/k_3)$ $(i = 1, 2, 3)$. Function $\phi(\xi_i, \tau)$ is the unsteady solution that satisfies Eqs. (1a), $\phi|_{\xi_1=0} = 0$, $\phi|_{\xi_3=1} = 0$, Eqs. (1c)–(1f). The initial condition at $\tau = 0$ requires

$$\phi(\xi_i, 0) = \psi(\xi_i) \quad i = 1, 2, 3 \tag{4}$$

We obtain the full series solution for the unsteady part by performing separation of variables as (the superscripts of the X_n are not powers):

$$\phi = \sum_{n=0}^{\infty} A_n e^{-\lambda_n^2\delta_1\tau} X_n^1(\xi_1) \quad \text{in } \xi_1 \in (0,1), \tag{5}$$

$$\phi = \Delta_1 \sum_{n=0}^{\infty} A_n e^{-\lambda_n^2\delta_1\tau} X_n^2(\xi_1) \quad \text{in } \xi_2 \in (0,1), \tag{6}$$

$$\phi = \Delta_2 \sum_{n=0}^{\infty} A_n e^{-\lambda_n^2\delta_1\tau} X_n^3(\xi_3) \quad \text{in } \xi_3 \in (0,1). \tag{7}$$

The eigenvalues λ_n satisfy the functional relationship (see [4] for similar results)

$$\tan(\lambda_n) = -(\Delta_1 \tan(\mu_{1n}) + \Delta_2 \tan(\mu_{2n}))/(1 - \Delta_2/\Delta_1 \times \tan(\mu_{1n}) \tan(\mu_{2n})). \quad (8)$$

The eigenfunctions are given by (see also [4]):

$$X_n^1(\xi_1) = \sin(\lambda_n \xi_1), \quad (9)$$

$$X_n^2(\xi_2) = \alpha_n \sin(\mu_{1n}\xi_2) + \beta_n \cos(\mu_{1n}\xi_2), \quad (10)$$

$$X_n^3(\xi_3) = \overline{\alpha}_n \sin(\mu_{2n}\xi_3) + \overline{\beta}_n \cos(\mu_{2n}\xi_3). \quad (11)$$

where the parameters can be calculated from: $\mu_{1n} = \sqrt{\delta_1/\delta_2}\lambda_n$, $\mu_{2n} = \sqrt{\delta_1/\delta_3}\lambda_n$,
$\Delta_1 = \kappa_1/\kappa_2\sqrt{\delta_2/\delta_1}$, $\Delta_2 = \kappa_1/\kappa_3\sqrt{\delta_3/\delta_1}$,
$\alpha_n = \cos(\lambda_n)$, $\beta_n = \sin(\lambda_n)/\Delta_1$,

$\overline{\alpha}_n = \cos(\lambda_n)\cos(\mu_{1n}) - \sin(\lambda_n)\sin(\mu_{1n})/\Delta_1$, and
$\overline{\beta}_n = \cos(\lambda_n)\sin(\mu_{1n})\Delta_1/\Delta_2 + \sin(\lambda_n)\cos(\mu_{1n})/\Delta_2$.

The global or 'natural' [4] orthogonality condition is

$$\eta_1 \int_0^1 X_n^1 X_m^1 \,d\xi_1 + \Delta_1 \int_0^1 X_n^2 X_m^2 \,d\xi_2 + \eta_2 \Delta_2 \int_0^1 X_n^3 X_m^3 \,d\xi_3 = \begin{cases} 0; & n \neq m \\ N_m; & n = m \end{cases}. \quad (12)$$

By a detailed derivation, we must have $\eta_1 = \sqrt{\delta_2/\delta_1}$ and $\eta_2 = \sqrt{\delta_2/\delta_3}$, so that when $n \neq m$,

$$\eta_1 \int_0^1 X_n^1 X_m^1 \,d\xi_1 + \Delta_1 \int_0^1 X_n^2 X_m^2 \,d\xi_2 + \eta_2 \Delta_2 \int_0^1 X_n^3 X_m^3 \,d\xi_3 = 0.$$

The orthogonality conditions can then be written in the following simple, symmetric form:

$$\kappa_2\kappa_3 \int_0^1 X_n^1 X_m^1 \,d\xi_1 + \kappa_1\kappa_3 \int_0^1 X_n^2 X_m^2 \,d\xi_2 + \kappa_1\kappa_2 \int_0^1 X_n^3 X_m^3 \,d\xi_3 = \begin{cases} 0; & n \neq m \\ M_m; & n = m \end{cases}. \quad (13)$$

Here $M_m = \kappa_2\kappa_3/2 + \kappa_1\kappa_3(\cos^2(\lambda_m) + \sin^2(\lambda_m)/(\Delta_1^2))/2 + \kappa_1 k_2(\overline{\alpha}_m^2 + \overline{\beta}_m^2)/2$ and $A_m = \kappa_2\kappa_3/(\lambda_m M_m)$. The expressions for A_m and M_m are used in Eqs. (5)–(7).

4. Numerical verification and comparison

We choose the following values for the problem parameters: $\delta_1 = 1.0$, $\kappa_1 = 1.0$ W/(m^2 K), $\delta_2 = 3.0$, $\kappa_2 = 0.1$ W/(m^2 K), $\delta_3 = 1.0$, $\kappa_3 = 1.0$ W/(m^2 K), i.e. $\kappa_1/\kappa_2 = 10$ and $\kappa_2/\kappa_3 = 0.1$.

The Newton iteration procedure outlined in the Appendix gives the eigenvalues of the problem, i.e. the roots of Eq. (8). The pattern of their variation is shown in Fig. 2. The first 20 eigenvalues are also given in Table 1.

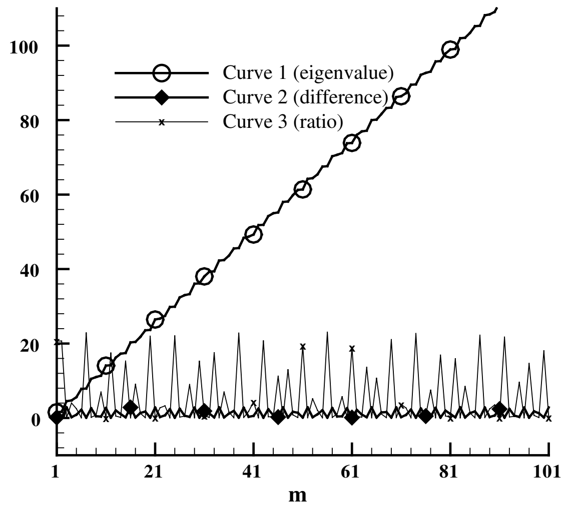

Fig. 2. Plot of λ_m, $\lambda_m - \lambda_{m-1}$, $\frac{\lambda_{m+1}-\lambda_m}{\lambda_m-\lambda_{m-1}}$ vs. m showing the variation of the eigenvalues.

Table 1
The first 20 eigenvalues, their differences and the ratios of successive differences

m	λ_m	$\lambda_m - \lambda_{m-1}$	$\frac{\lambda_{m+1}-\lambda_m}{\lambda_m-\lambda_{m-1}}$
1	1.5432e+00	–	–
2	1.6799e+00	1.3670e−01	2.0678e+01
3	4.5066e+00	2.8267e+00	7.7087e−02
4	4.7245e+00	2.1790e−01	4.0319e+00
5	5.6031e+00	8.7855e−01	2.5087e+00
6	7.8070e+00	2.2040e+00	5.1384e−02
7	7.9203e+00	1.1325e−01	2.2846e+01
8	1.0508e+01	2.5872e+00	1.8792e−01
9	1.0994e+01	4.8619e−01	7.8923e−01
10	1.1377e+01	3.8372e−01	6.9958e+00
11	1.4062e+01	2.6844e+00	4.3387e−02
12	1.4178e+01	1.1647e−01	1.7475e+01
13	1.6214e+01	2.0352e+00	5.1553e−01
14	1.7263e+01	1.0492e+00	1.7723e−01
15	1.7449e+01	1.8595e−01	1.5299e+01
16	2.0293e+01	2.8448e+00	5.2749e−02
17	2.0444e+01	1.5006e−01	9.0633e+00
18	2.1804e+01	1.3600e+00	1.2692e+00
19	2.3530e+01	1.7262e+00	7.3562e−02
20	2.3657e+01	1.2698e−01	2.1993e+01

We implemented both the eigenfunction expansion and finite difference methods for this numerical example. We analyzed the error between the expansion solution (θ_e) and finite difference solution (θ_{fd}) in the normalized L^2 norm, $\|\theta_e - \theta_{fd}\|_{L^2}/\|\theta_e\|_{L^2}$. We found that they are identical to within 0.15% at $\tau = 0.1$, 0.42% at $\tau = 0.5$, 0.21% at $\tau = 1.0$, 0.035% at $\tau = 2.0$, and essentially equal at $\tau = 100$. Fig. 3a shows the eigenfunction

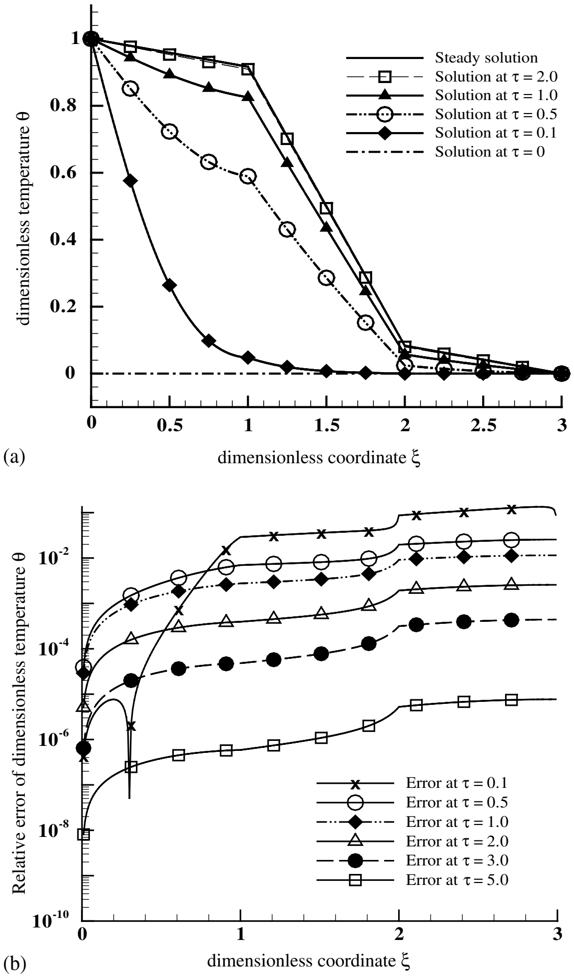

Fig. 3. (a) Solutions using the eigenfunction expansion and finite difference methods are graphically indistinguishable. The interfaces are located at $\xi = 1$, $\xi = 2$. The three-layer medium is confined to $0 \leqslant \xi \leqslant 3$. (b) Relative errors $|\theta_e - \theta_{fd}|/|\theta_e|$ at each point ξ at the specified time τ. Note the decrease as τ increases.

$$\tan(\lambda_l) = -\frac{\sum_{i=1}^{n-1} \Delta_i \tan\left(\sqrt{\frac{\delta_1}{\delta_{i+1}}}\lambda_l\right) - \sum_{i=3}^{n-1}\sum_{j=2}^{i-1}\sum_{k=1}^{j-1} \frac{\Delta_k \Delta_i}{\Delta_j} \tan\left(\sqrt{\frac{\delta_1}{\delta_{k+1}}}\lambda_l\right)\tan\left(\sqrt{\frac{\delta_1}{\delta_{j+1}}}\lambda_l\right)\tan\left(\sqrt{\frac{\delta_1}{\delta_{i+1}}}\lambda_l\right) + \cdots}{1 - \sum_{i=2}^{n-1}\sum_{j=1}^{i-1} \frac{\Delta_i}{\Delta_j} \tan\left(\sqrt{\frac{\delta_1}{\delta_{j+1}}}\lambda_l\right)\tan\left(\sqrt{\frac{\delta_1}{\delta_{i+1}}}\lambda_l\right) + \sum\sum\sum\sum - \cdots} \quad (16)$$

solutions and the finite difference numerical solutions. Fig. 3b gives the relative errors versus the dimensionless coordinates. We see that the solutions agree well with each other.

We also examined the number of terms required for an accurate description when the eigenfunction expansion method is implemented. Only a few terms (no more than three) are needed for the "large" time $\tau = 1.0$, but more terms (approximately 15) must be included to achieve the same accuracy for $\tau = 0.01$. Producing accurate solutions with few terms is important if the expression is actually to be used in heat transfer calculations [9].

5. Conjectured solution for an *n*-layer slab

Consider a composite slab consisting of n parallel layers as shown in Fig. 4. The non-dimensional temperature θ satisfies Eq. (1a) in the ith region. The boundary conditions and the initial conditions are the same as the case of 2-layer (or 3-layer) slab. We assume also that the interfaces satisfy conditions of perfect thermal contact.

Based on the results for the 2-layer and 3-layer cases, we write the following conjectured solution for the n-layer problem:

$$\left.\begin{array}{ll} \theta = 1 - (\Delta\theta)_1\xi_1 - \phi & \text{in } \xi_1 \in (0,1) \\ \theta = 1 - (\Delta\theta)_1 - (\Delta\theta)_2\xi_2 - \phi & \text{in } \xi_2 \in (0,1) \\ \vdots & \\ \theta = 1 - (\Delta\theta)_1 - \cdots - (\Delta\theta)_n\xi_n - \phi & \text{in } \xi_n \in (0,1) \end{array}\right\} \quad (14)$$

$$\begin{aligned} \phi &= \sum_{l=0}^{\infty} A_l e^{-\lambda_l^2\delta_1\tau} X_l^1(\xi_1) \quad \text{in } \xi_1 \in (0,1), \\ \phi &= \Delta_1 \sum_{l=0}^{\infty} A_l e^{-\lambda_l^2\delta_1\tau} X_l^2(\xi_2) \quad \text{in } \xi_2 \in (0,1), \\ \phi &= \Delta_2 \sum_{l=0}^{\infty} A_l e^{-\lambda_l^2\delta_1\tau} X_l^3(\xi_3) \quad \text{in } \xi_3 \in (0,1), \\ &\vdots \\ \phi &= \Delta_{n-1} \sum_{l=0}^{\infty} A_l e^{-\lambda_l^2\delta_1\tau} X_l^n(\xi_n) \quad \text{in } \xi_n \in (0,1), \end{aligned} \quad (15)$$

where the eigenvalues λ_l satisfy the function:

The global orthogonality condition is:

$$\prod_{j\neq 1}\kappa_j \int_0^1 X_p^1 X_q^1 \,d\xi_1 + \prod_{j\neq 2}\kappa_j \int_0^1 X_p^2 X_q^2 \,d\xi_2 + \cdots + \prod_{j\neq n}\kappa_j \int_0^1 X_p^n X_q^n \,d\xi_n = \begin{cases} 0; & p \neq q \\ M_q; & p = q. \end{cases} \quad (17)$$

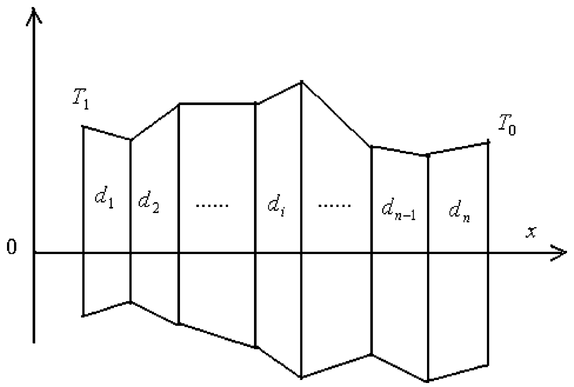

Fig. 4. A composite slab with n distinct layers.

The coefficients A_l in Eq. (15) are given by:

$$A_l = \prod_{j \neq l} \kappa_j / (\lambda_l M_l). \quad (18)$$

6. Conclusion

The eigenfunction expansion method is shown to provide an easily calculated exact solution that requires especially few terms for large times. The advantage of this solution method may arise, for example, when one of the slabs has thermophysical properties radically different from those of the other two, or one layer (e.g., the sample) is much narrower than the other two layers. In cases such as these a numerical solution undertaken without a prior transformation of coordinates may become inaccurate. The investigation of such special limiting cases remains a subject for future research.

Acknowledgements

The second author wishes to acknowledge discussions of this and other heat conduction problems with Professor J.V. Beck (MSU, emeritus). His research was partially supported by NASA Microgravity Combustion Research Contract # NCC3-662, monitored by Dr. Fletcher Miller.

Appendix A

We describe the procedure for finding eigenvalues by the Newton iterative method mentioned in Section 4. We assume that the analytic solution has the form given by Eqs. (5)–(7). The eigenfunctions $X_n^j(\xi)$, $(j = 1, 2, 3)$ are bounded. The eigenvalues λ_n satisfy $F(\lambda) = 0$. When $\delta\tau$ is not small, $e^{-\lambda^2 \delta\tau}$ decreases rapidly with an increase of $|\lambda|$, therefore we only need to consider λ's ($|\lambda| \leqslant \Lambda$, for a sufficiently large $\Lambda > 0$) to ensure the required accuracy. By symmetry, we only need to find all λ's ($\lambda \in [0, \Lambda]$) to accomplish the implementation. The interval $[0, \Lambda]$ is divided into N non-overlapping subintervals, i.e. $[0, \Lambda] = \bigcup_{i=1}^{N} [a_i, a_{i+1}]$ such that in $[a_i, a_{i+1}]$ $(i = 1, 2, \ldots N)$ $F(\lambda)$ is monotonous. Then we can apply the Newton iterative method to find the corresponding $\lambda(\lambda \in [a_i, a_{i+1}])$ such that $F(\lambda) = 0$, where a_i may be taken as the initial value of the iteration.

References

[1] M.N. Özisik, Heat Conduction, second ed., John Wiley & Sons, Inc., New York, 1993.
[2] M.D. Mikhailov, M.N. Özisik, N.L. Vulchanov, Diffusion in composite layers with automatic solution of the eigenvalue problem, Int. J. Heat Mass Transfer 26 (8) (1983) 1131–1141.
[3] S.C. Huang, Y.P. Chang, Heat conduction in unsteady, periodic and steady states in laminated composites, J. Heat Transfer 102 (4) (1980) 742–748.
[4] F. de Monte, Transient heat conduction in one-dimensional composite slab, A 'natural' analytical approach, Int. J. Heat Mass Transfer 43 (2000) 3607–3619.
[5] F. de Monte, An analytic approach to the unsteady heat conduction processes in one-dimensional composite media, Int. J. Heat Mass Transfer 45 (2002) 1333–1343.
[6] L. Yan, A. Haji-Sheikh, J.V. Beck, Thermal characteristics of two-layered bodies with imbedded thin-film heat source, J. Electron. Packaging 115 (1993) 276–283.
[7] I. Glassman, Combustion, third ed., Academic Press, San Diego, CA, 1996.
[8] A. Haji-Sheikh, J.V. Beck, D. Agonafer, Steady-state heat conduction in multi-layer bodies, Int. J. Heat Mass Transfer 46 (2003) 2363–2379.
[9] D.H.Y. Yen, J.V. Beck, R.L. McMasters, D.E. Amos, Solution of an initial-boundary value problem for heat conduction in a parallelepiped by time partitioning, Int. J. Heat Mass Transfer 45 (2002) 4267–4279.